Electromagnetic waves

Prepared by

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Vector Analysis

SCALARS AND VECTORS

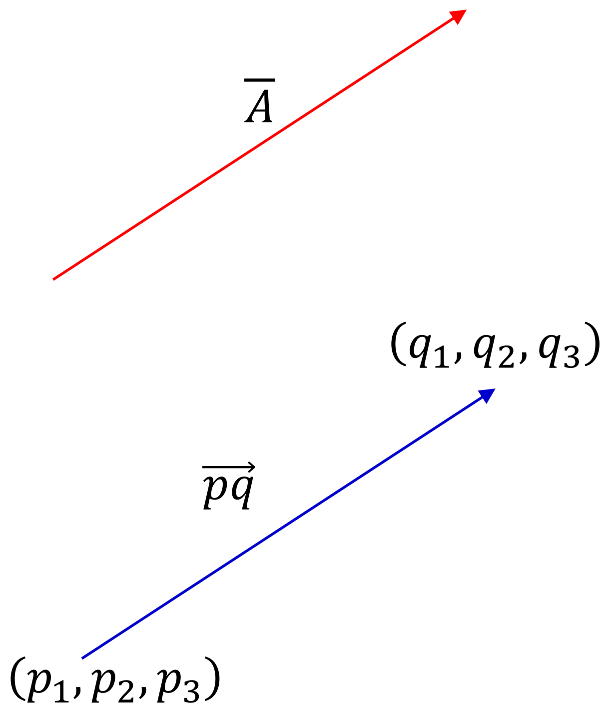
- The term ***scalar*** refers to a quantity whose value may be represented by a single (positive or negative) real number. Like distance, temperature, mass, density, pressure, and volume.
- A ***vector*** has both a magnitude and a direction in space. Like Force, velocity, and acceleration.
- Our work will mainly concern scalar and vector *fields.*
- A field (scalar or vector) may be defined mathematically as some function that connects an arbitrary origin to a general point in space.
- The value of a field varies in general with both position and time.

Vector Analysis

VECTOR ALGEBRA

- A vector is determined by its length and direction. They are usually denoted with letters with arrows on the top $\overline{A}$ or in bold letter **A**.
- If we are given two points in the space (p1, p2, p3) and (q1, q2, q3) then we can compute the vector that goes from p to q as follows:

$$\vec{pq} = [q_1 - p_1, q_2 - p_2, q_3 - p_3]$$



Vector Analysis

COORDINATE SYSTEMS

- ***RECTANGULAR or Cartesian***
- ***CYLINDRICAL***
- ***SPHERICAL***

Choice is based on symmetry of problem

<u>Examples:</u>

Sheets - RECTANGULAR

Wires/Cables - CYLINDRICAL

Spheres - SPHERICAL

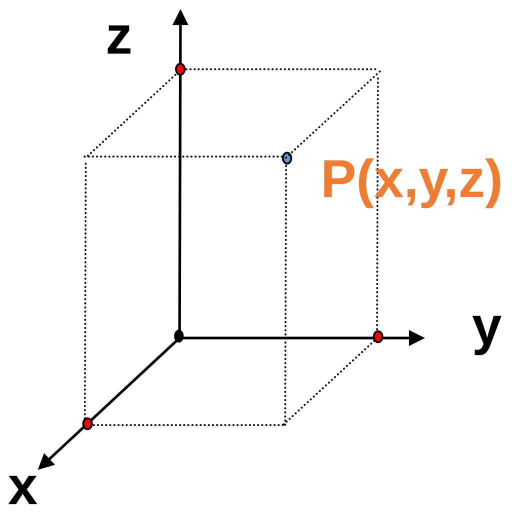
Cartesian Coordinates Or Rectangular Coordinates

P (x, y, z)

$$-\infty < x < \infty$$

$$-\infty < y < \infty$$

$$-\infty < z < \infty$$



A vector A in Cartesian coordinates can be written as

$$(A_x, A_y, A_z) \quad \text{or} \quad A_x a_x + A_y a_y + A_z a_z$$

where a_x,a_y and a_z are unit vectors along x, y and z-directions.

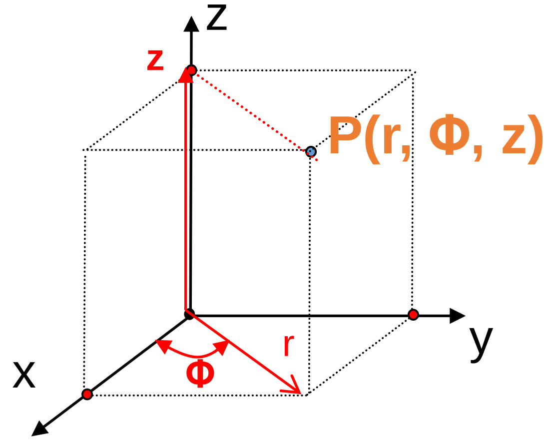
Cylindrical Coordinates

P (r, Φ, z)

$$0 \le r < \infty$$

$$0 \le \phi < 2\pi$$

$$-\infty < z < \infty$$



A vector A in Cylindrical coordinates can be written as

$$(A_r, A_\phi, A_z) \quad \text{or} \quad A_r a_r + A_\phi a_\phi + A_z a_z$$

where a_r,a_Φ and a_z are unit vectors along r, Φ and z-directions.

x= r cos Φ, y=r sin Φ,   z=z

$$r = \sqrt{x^2 + y^2}, \phi = \tan^{-1}\frac{y}{x}, z = z$$

The relationships between (a_x,a_y, a_z) and (a_r,a_Φ, a_z)are

$$a_x = \cos\phi a_r - \sin\phi a_\phi$$

$$a_y = \sin\phi a_r - \cos\phi a_\phi$$

$$a_z = a_z$$

or

$$a_r = \cos\phi a_x + \sin\phi a_y$$

$$a_\phi = -\sin\phi a_x + \cos\phi a_y$$

$$a_z = a_z$$

Then the relationships between (A_x,A_y, A_z) and (A_r, A_Φ, A_z)are

$$A = (A_x \cos\phi + A_y \sin\phi) a_r + (-A_x \sin\phi + A_y \cos\phi) a_\phi + A_z a_z$$

$$A_r = A_x \cos\phi + A_y \sin\phi$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$A_z = A_z$$

In matrix form we can write

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

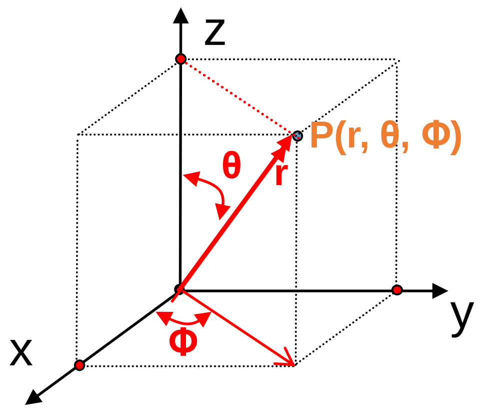
Spherical Coordinates

P (r, θ, Φ)

$$0 \le r < \infty$$

$$0 \le \theta \le \pi$$

$$0 \le \phi < 2\pi$$



A vector A in Spherical coordinates can be written as

$$(A_r, A_\theta, A_\phi) \quad \text{or} \quad A_r a_r + A_\theta a_\theta + A_\phi a_\phi$$

where a_r, a_θ, and a_Φ are unit vectors along r, θ, and Φ-directions.

x=r sin θ cos Φ, y=r sin θ sin Φ, Z=r cos θ

$$r = \sqrt{x^2 + y^2 + z^2}, \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \phi = \tan^{-1}\frac{y}{x}$$

The relationships between (a_x,a_y, a_z) and (a_r,a_θ,a_Φ)are

$$a_x = \sin\theta\cos\phi a_r + \cos\theta\cos\phi a_\theta - \sin\phi a_\phi$$

$$a_y = \sin\theta\sin\phi a_r + \cos\theta\sin\phi a_\theta + \cos\phi a_\phi$$

$$a_z = \cos\theta a_r - \sin\theta a_\theta$$

or

$$a_r = \sin\theta\cos\phi a_x + \sin\theta\sin\phi a_y + \cos\theta a_z$$

$$a_\theta = \cos\theta\cos\phi a_x + \cos\theta\sin\phi a_y - \sin\theta a_z$$

$$a_\phi = -\sin\phi a_x + \cos\phi a_y$$

Then the relationships between (A_x,A_y, A_z) and (A_r, A_θ,and A_Φ)are

$$A = (A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta)a_r$$

$$+(A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta)a_\theta$$

$$+(-A_x \sin\phi + A_y \cos\phi)a_\phi$$

$$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$$

$$A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

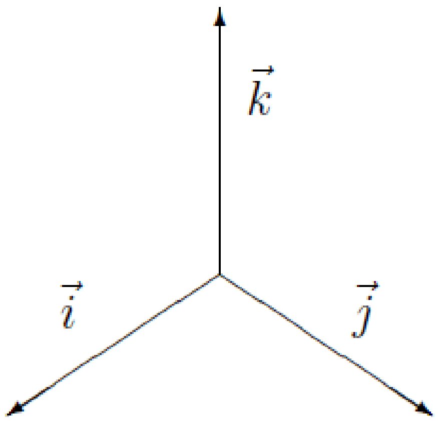
In matrix form we can write

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Vector Analysis

VECTOR ALGEBRA

- The three principal directions (unit vectors, vectors of length one) in the space are



$$\bar{i} = [1,0,0], \bar{j} = [0,1,0], \bar{k} = [0,0,1]$$

- The length (magnitude) of a vector with coordinates $[A_x, A_y, A_z]$ is

$$|\bar{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2},$$

With unit vector $\overline{a_A} = \frac{\bar{A}}{|\bar{A}|}$

Vector Analysis

VECTOR ALGEBRA

- If we have two vectors $\overline{A}$ and $\overline{B}$

$$\overline{A}+\overline{B}=(A_x+B_x)\hat{x}+\left(A_y+B_y\right)\hat{y}+(A_z+B_z)\hat{z}$$

or

$$\overline{A}+\overline{B}=(A_x+B_x)\hat{\imath}+\left(A_y+B_y\right)\hat{\jmath}+(A_z+B_z)\hat{k}$$

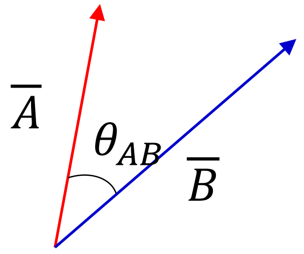
$$\overline{A}-\overline{B}=(A_x-B_x)\hat{x}+\left(A_y-B_y\right)\hat{y}+(A_z-B_z)\hat{z}$$

$$\beta(\overline{A})=\beta A_x\hat{x}+\beta A_y\hat{y}+\beta A_z\hat{z}$$

Vector Analysis

- ***Dot product***, or ***scalar product***

$$\overline{A} \cdot \overline{B} = |\overline{A}||\overline{B}| \cos \theta_{AB}$$
$$\overline{A} \cdot \overline{B} = A_x B_x + A_y B_y + A_z B_z$$

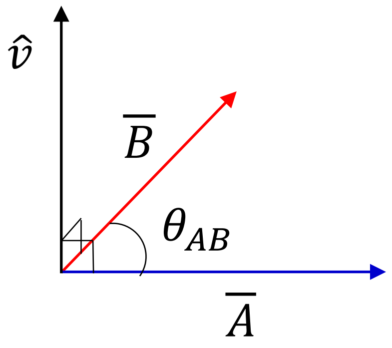


- ***Vector product (cross-product)***

It is denoted by $\overline{V} = \overline{A} x \overline{B}$ where

$$\overline{A} x \overline{B} = |\overline{A}||\overline{B}| \sin \theta_{AB} \, \hat{v}$$

$$\overline{A} x \overline{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



Gradient, Divergence and Curl

The Del Operator

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$$

- Gradient of a scalar function is a vector quantity. $\nabla f \longrightarrow$ Vector
- Divergence of a vector is a scalar quantity. $\nabla . A \longrightarrow$ Scalar
- Curl of a vector is a vector quantity. $\nabla \times A \longrightarrow$ Vector

Gradient, Divergence and Curl

Gradient of a Scalar

The gradient of a scalar field V is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

$$\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

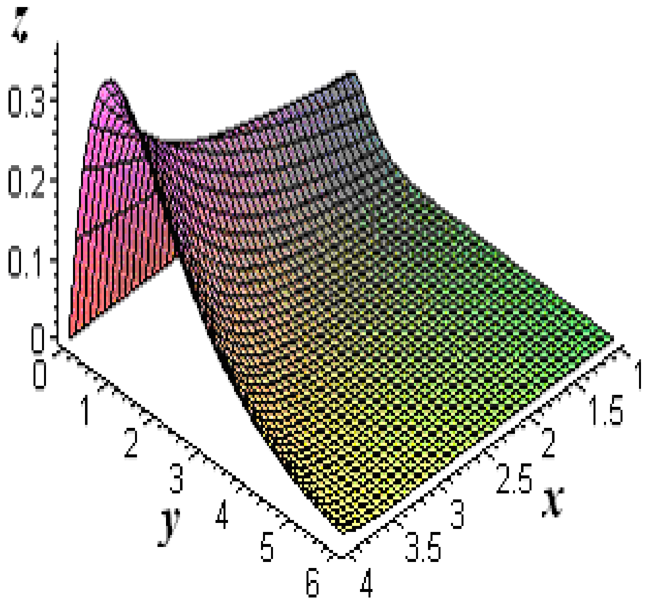
Gradient, Divergence and Curl

PHYSICAL INTERPRETATION OF GRADIENT

- One is given in terms of the graph of some function z = f(x, y), where the graph is a surface whose points have variable heights over the x y – plane.
- An illustration is given below.

  If, say, we place a marble at some point

  (x, y) on this graph with zero initial force, its motion will trace out a path on the surface, and in fact it will choose the direction of steepest descent.



- This direction of steepest descent is given by the negative of the gradient of f. One takes the negative direction because the height is decreasing rather than increasing.

Gradient, Divergence and Curl

Gradient of a scalar field important relations

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* $\nabla(V + U) = \nabla V + \nabla U$
* $\nabla(VU) = V\nabla U + U\nabla V$

Gradient, Divergence and Curl

Find gradient of this scalar field:

$$V = e^{-z} \sin 2x \cosh y$$

Answer

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

$$= 2e^{-z} \cos 2x \cosh y \mathbf{a}_x + e^{-z} \sin 2x \sinh y \mathbf{a}_y$$

$$- e^{-z} \sin 2x \cosh y \mathbf{a}_z$$

Gradient, Divergence and Curl

Divergence of a Vector

The divergence of A at a given point P is the outward flux per unit volume as the volume shrinks about P.

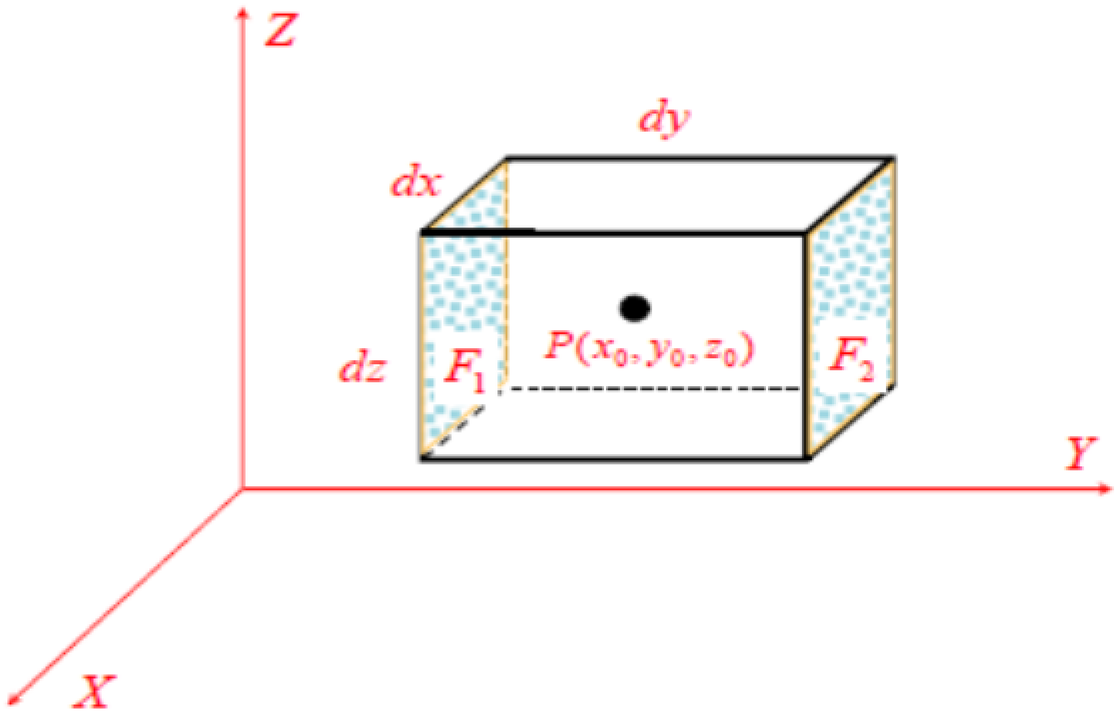
$$divA = \nabla . A = \lim_{\Delta v \to 0} \frac{\oint_S A.dS}{\Delta v}$$

For Cartesian coordinate:

$$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Gradient, Divergence and Curl

Divergence of a vector in Cartesian coordinates



Gradient, Divergence and Curl

Divergence of a vector in Cartesian coordinates

- *To evaluate the divergence of a vector field* $\vec{A}$ *at point* $P(x_0, y_0, z_0)$ *first construct a differential volume around point P*
- *The closed surface integral of* $\vec{A}$ *is obtained as*

$$\oint_S \vec{A} \cdot \overrightarrow{dS} = \left(\int_{FRONT} + \int_{BACK} + \int_{LEFT} + \int_{RIGHT} + \int_{TOP} + \int_{BOTTOM} \right)$$

- *A three dimensional Taylors series expansion of* A_x *about P is*

$$A_x(x,y,z) = A_x(x_0,y_0,z_0) + (x - x_0)\left.\frac{\partial A_x}{\partial x}\right|_P + (y - y_0)\left.\frac{\partial A_x}{\partial y}\right|_P + (z - z_0)\left.\frac{\partial A_x}{\partial z}\right|_P$$

$$+ \text{ higher order terms}$$

Gradient, Divergence and Curl

Divergence of a vector in Cartesian coordinates

$$\text{For the front side } x = x_0 + \frac{dx}{2}, \vec{A} = A_x \hat{a}_x, \overrightarrow{dS} = dydz\hat{a}_x$$

$$\int_{FRONT} \vec{A} \cdot \overrightarrow{dS} = \left(A_x(x_0, y_0, z_0) + \frac{dx}{2} \frac{\partial A_x}{\partial x}\bigg|_P \right) dydz + \text{higher order terms}$$

$$\text{For the back side } x = x_0 - \frac{dx}{2}, \vec{A} = A_x(-\hat{a}_x), \overrightarrow{dS} = dydz(-\hat{a}_x)$$

$$\int_{BACK} \vec{A} \cdot \overrightarrow{dS} = -\left(A_x(x_0, y_0, z_0) - \frac{dx}{2} \frac{\partial A_x}{\partial x}\bigg|_P \right) dydz + \text{higher order terms}$$

$$\int_{FRONT} \vec{A} \cdot \overrightarrow{dS} + \int_{BACK} \vec{A} \cdot \overrightarrow{dS} = dxdydz \frac{\partial A_x}{\partial x}\bigg|_P + \text{higher order terms}$$

Gradient, Divergence and Curl

Divergence of a vector in Cartesian coordinates

Similarly

$$\int_{LEFT} \vec{A}\cdot\overrightarrow{dS} + \int_{RIGHT} \vec{A}\cdot\overrightarrow{dS} = dxdydz\left.\frac{\partial A_y}{\partial y}\right|_P + \text{higher order terms}$$

$$\int_{TOP} \vec{A}\cdot\overrightarrow{dS} + \int_{BOTTOM} \vec{A}\cdot\overrightarrow{dS} = dxdydz\left.\frac{\partial A_z}{\partial z}\right|_P + \text{higher order terms}$$

$$\oint_S \vec{A}\cdot\overrightarrow{dS} = dxdydz\left.\frac{\partial A_x}{\partial x}\right|_P + dxdydz\left.\frac{\partial A_y}{\partial y}\right|_P + dxdydz\left.\frac{\partial A_z}{\partial z}\right|_P + \text{higher order terms}$$

$$\oint_S \vec{A}\cdot\overrightarrow{dS} = \left.\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right|_P \Delta v + \text{higher order terms}$$

Substituting in $\lim_{\partial v\to 0}\frac{\oint_S \vec{A}\cdot\overrightarrow{dS}}{\Delta v}$

Gradient, Divergence and Curl

Divergence of a vector in Cartesian coordinates

$$\lim_{\delta v \to 0} \frac{\oint_S \vec{A} \cdot \vec{dS}}{\Delta v} = \lim_{\delta v \to 0} \frac{\left.\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right|_P \Delta v}{\Delta v} \quad = \left.\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right|_P$$

Since higher order terms vanish as $\Delta v \to 0$

Divergence of $\vec{A}$ *at* $P(x_0, y_0, z_0)$ *in Cartesian coordinates is*

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

Find divergence of these vectors:

$$P = x^2 yz\mathbf{a}_x + xz\mathbf{a}_z$$

Answer

$$\nabla \bullet \mathbf{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

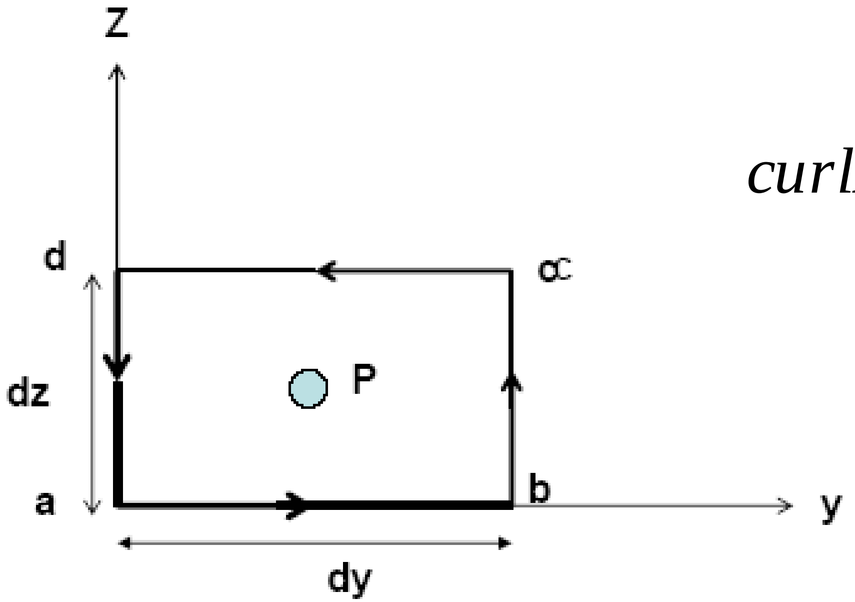
$$= \frac{\partial}{\partial x}\left(x^2 yz\right) + \frac{\partial}{\partial y}\left(0\right) + \frac{\partial}{\partial z}\left(xz\right)$$

$$= 2xyz + x$$

Gradient, Divergence and Curl

Curl of a vector

The curl of A is an axial vector whose magnitude is the maximum circulation of A per unit area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.



$$curl A = \nabla \times A = \left(\lim_{\Delta s \to 0} \frac{\oint_L A.dl}{\Delta S} \right)_{max} a_n$$

$$\text{Where, } \oint_s \mathbf{A} \bullet dl = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \right) \mathbf{A} \bullet dl$$

ΔS is the area bounded by the curve L and a_n is the unit vector normal to the surface ΔS

Gradient, Divergence and Curl

CURL OF A VECTOR (Cont'd)

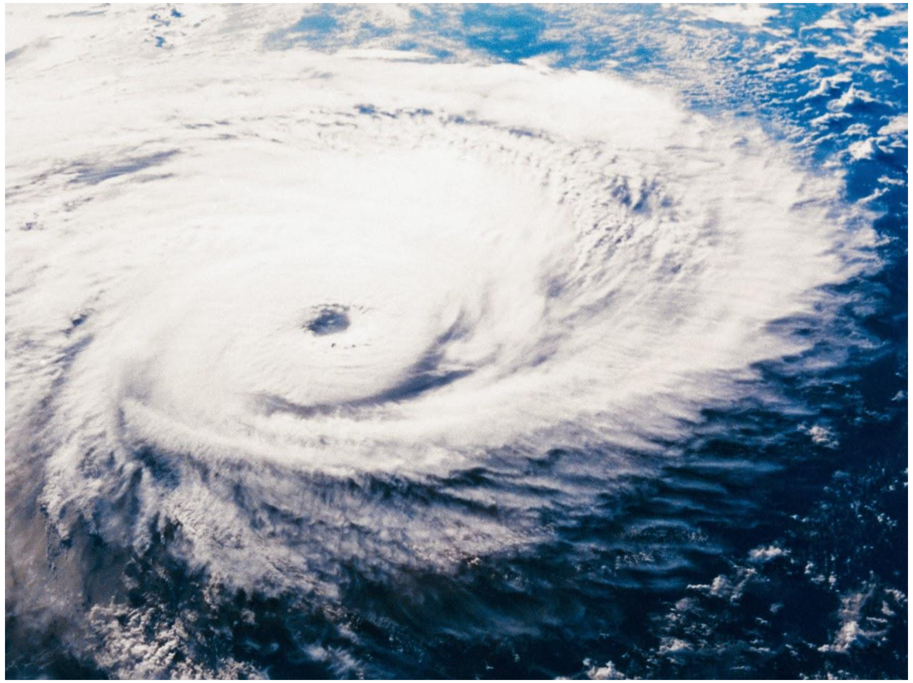
For Cartesian coordinate:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathbf{a}_x - \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right]\mathbf{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\mathbf{a}_z$$

Gradient, Divergence and Curl

CURL OF A VECTOR (Cont'd)



Gradient, Divergence and Curl

Divergence or Gauss' Theorem

The divergence theorem states that the total outward flux of a vector field A through the closed surface S is the same as the volume integral of the divergence of A.

$$\oint A.dS = \int_V \nabla.A dv$$

Gradient, Divergence and Curl

Stokes' Theorem

Stokes's theorem states that the circulation of a vector field A around a closed path L is equal to the surface integral of the curl of A over the open surface S bounded by L, provided A and $\nabla \times A$ *are continuous on S*

$$\oint_L A.dl = \int_S (\nabla \times A).dS$$